# Continuity:

Handwavy understanding is, it's about being able to draw graph continuously without lifting the pen/pencil from the paper.

**Def$^n$**:- A function $f(x)$ is continuous at $x=a$ if & only if

$$\left.\begin{array}{l} \text{(i) } f(a) \text{ is defined} \\ \text{(ii) } \lim_{x\to a} f(x) \text{ exists} \\ \text{(iii) } \lim_{x\to a} f(x) = f(a) \end{array}\right\} \text{satisfies.}$$

(*) If it fails to be continuous, we say it's discontinuous at $x=a$.

(*) $f$ is continuous on an interval if $f$ is continuous at every point $a$ in that interval.

## Determining Continuity at a point:

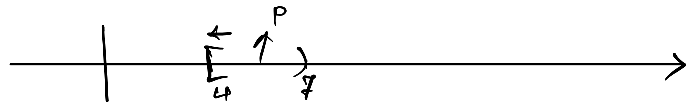
$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Then $f(0)=1$ & we have seen $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Therefore, $f(x)$ is continuous at $x=0$

Determing Continuity over an Interval:

$f(x) = \sqrt{x-4}$ , $x$ is in $[4, 7)$




<u>Note</u>: $[4, 7)$ means domain has the point $x=4$ in it, but not 7.

So we need to check for two cases:

<u>Case 1</u>: Continuity at $x=4$

We need to check for $\lim_{x \to 4^+} f(x)$ & $f(4)$ only.

Now, $f(4) = \sqrt{4-4} = \sqrt{0} = 0$

& $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$

So, $\lim_{x \to 4^+} f(x)$ exists & $\lim_{x \to 4} f(x) = f(4)$.

$\Rightarrow f(x)$ is continuous at $x=4$.

<u>Case 2</u>: Continuity at any interior point $x = t$ in $[4, 7)$

Now, $\lim_{x \to t^-} f(x) = \lim_{x \to t^-} \sqrt{x-4} = \sqrt{t-4}$

$\lim_{x \to t^+} f(x) = \lim_{x \to t^-} \sqrt{x-4} = \sqrt{t-4}$

$\Big\}\ \lim_{x \to t} f(x) = \sqrt{t-4}$

& $f(t) = \sqrt{t-4}$ , so $f(x)$ is Cont. at any interior point of $[4, 7)$

## Functions that are continuous everywhere:

① $f(x) =$ polynomial in $x$ $(x^5 + 7x + 5)$

② $f(x) = |x|$ ⇝ the absolute value function.

③ $f(x) =$ exponential functions $(e^x, e^{-(x+4)})$

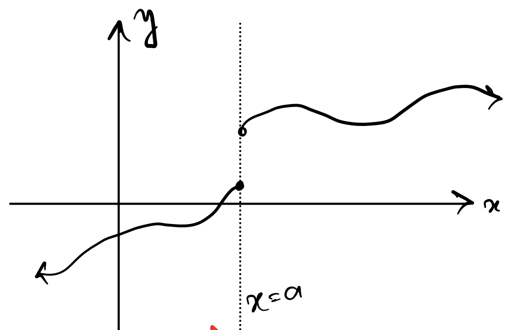
④ $f(x) = \sin x / \cos x$.

* $\tan x$, $\csc x$, $\sec x$, $\cot x$ are continuous on their respective domains.

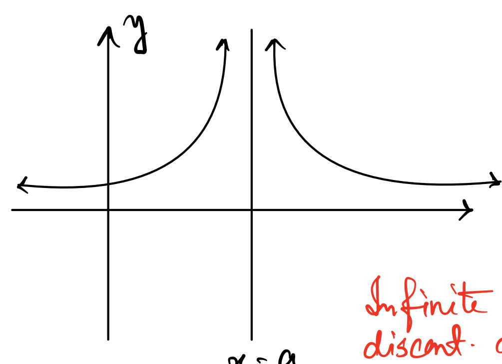
$\log_a x$, $\ln x$, $\sqrt{x}$, $\sqrt[n]{x}$, rational functions are also continuous on their respective domains.

## Discontinuties

Situation 1. $\lim_{x \to a} f(x)$ DNE.

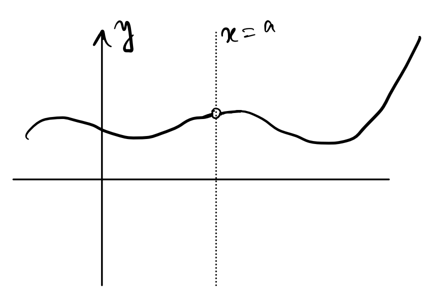



Jump Discont. at $x=a$



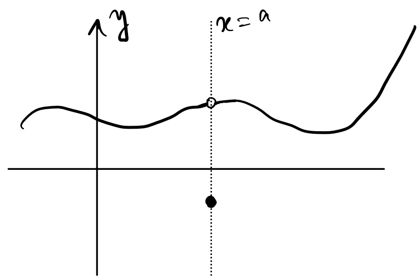

Infinite discont. at $x=a$

Situation 2. $f(a)$ DNE




Removable discontinuity at $x=a$

Situation 3. Both $\lim_{x \to a} f(x)$ & $f(a)$ exists, but not equal.




Removable discontinuity at $x=a$

So there are three type of Discontinuity:

① Infinite, when $\lim_{x \to a^{\pm}} f(x) = \pm\infty$

② Removable, define $f(a)$ intelligently.

③ Jump, when $\lim_{x \to a^{+}} f(x) \neq \lim_{x \to a^{+}} f(x)$

Eg. $f(x) = \dfrac{x^2 - 2x}{x^2 - 4} = \dfrac{x(x-2)}{(x+2)(x-2)}$

Domain $= \{x \mid x \neq -2, 2\}$

At $x = -2$,

$\lim_{x \to -2} f(x) = \pm\infty \Rightarrow$ Infinite discont. at $x=-2$

At $x=2$, $\lim_{x \to 2} f(x) = \frac{1}{2}$, but it is not defined, so $f(2)$ DNE.

$\Rightarrow$ Removable discont at $x=2$

Note:- If $f(x)$ is cont at $x=a$, then the limits exists at $x=a$.

But if the limit exists at $x=a$, then it might or might not be continuous at $x=a$.